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A BOUNDARY VALUE PROBLEM FOR A FRACTIONAL ORDER ORDINARY LINEAR DIFFERENTIAL EQUATION WITH A CONSTANT COEFFICIENT*

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Abstract. In the paper a boundary condition is given according to the number of steps for obtaining higher order derivative in the boundary value problem for a fractional ordinary differential equations.

Keywords: fractional derivative, Liouville definition, Mittag-Leffler functions, invariants for fractional derivative.

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1. Introduction

As unlike an ordinary derivative, the fractional derivative is given in the global form (by an integral), sometimes we can get from the given equation such order derivative that the order of the obtained equation is an integer greater or close to the given order [1-3, 5, 6]

Writing integral expressions of all the terms of the differential equation with the exception of the principal term (the order of this term is entire), we get an integro-differential equation [4]. This time the number of conditions is taken equal to this higher order. Note, that conditions is taken equal to this higher order. Note that the obtained problem is not already the previous problem. It is appropriate to investigate the solution of the problem stated for this equation not changing the order of the given equation [6].

2. Problem statement

Let's consider the following boundary value problem

$$ly \equiv \sum_{k=0}^{m} a_k D^{\frac{k}{n}}(x) = 0,\ x \in (a,b),\ 0 \notin (a,b), \tag{1}$$

$$\sum_{k=0}^{m-1} \left\{ \alpha_{jk} D^{\frac{k}{n}} y(x)\Big|_{x=a} + \beta_{jk} D^{\frac{k}{n}} y(x)\Big|_{x=b} \right\} = \gamma_j,\ \ j=\overline{1,m}, \tag{2}$$

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where $a_k,\ k=\overline{0,m};\ \alpha_{jk},\ \beta_{jk}\ \ j=\overline{1,m},\ \ k=\overline{0,m-1};\ \gamma_j\ \ j=\overline{1,m}$ are the given constants n is the noted natural number. As the order of the derivative in the considered equation changes in the step $\frac{1}{n}$, we consider the following functions [1], [6]

$$h_{\frac{1}{n}}(x,\lambda)=\frac{x^{-1+\frac{1}{n}}}{\left(-1+\frac{1}{n}\right)!}+\frac{x^{-1+\frac{2}{n}}\lambda}{\left(-1+\frac{2}{n}\right)!}+\ldots=\sum_{s=0}^{m}\frac{x^{-1+\frac{s}{n}}\lambda^{s-1}}{\left(-1+\frac{s}{n}\right)!}, \tag{3}$$

where λ is an arbitrary constant. It is easy to see that

$$D^{\frac{1}{n}}h_{\frac{1}{n}}(x,\lambda)=D^{\frac{1}{n}}\frac{x^{-1+\frac{1}{n}}}{\left(-1+\frac{1}{n}\right)!}+D^{\frac{1}{n}}\frac{x^{-1+\frac{2}{n}}\lambda}{\left(-1+\frac{2}{n}\right)!}+D^{\frac{1}{n}}\frac{x^{-1+\frac{3}{n}}\lambda^2}{\left(-1+\frac{3}{n}\right)!}+\ldots=$$

$$=\frac{x^{-1}}{(-1)!}+\frac{x^{-1+\frac{1}{n}}\lambda}{\left(-1+\frac{1}{n}\right)!}+\frac{x^{-1+\frac{2}{n}}\lambda^2}{\left(-1+\frac{2}{n}\right)!}+\ldots=$$

$$=\lambda\left[\frac{x^{-1+\frac{1}{n}}}{\left(-1+\frac{1}{n}\right)!}+\frac{x^{-1+\frac{2}{n}}\lambda}{\left(-1+\frac{2}{n}\right)!}+\ldots\right]=\lambda h_{\frac{1}{n}}(x,\lambda). \tag{4}$$

Allowing for (4), substitute (3) in equation (1), and with respect to λ we get the following characteristic equation

$$\sum_{k=0}^{m}a_k\lambda^k=0. \tag{5}$$

Let the roots of characteristic equations different. Denote them by $\lambda_1,\ \lambda_2,\ldots,\lambda_m$.

Then the general solution of equation (1) will be in the following form

$$y(x,\lambda)=\sum_{s=1}^{m}C_s h_{\frac{1}{n}}(x,\lambda_s). \tag{6}$$

Define the arbitrary constants contained in this expression from the boundary conditions. For that substitute (6) in (2),

$$\sum_{s=1}^{m}C_s\sum_{k=0}^{m-1}\lambda_s^k\left[\alpha_{jk}h_{\frac{1}{n}}(a,\lambda_s)+\beta_{jk}h_{\frac{1}{n}}(b,\lambda_s)\right]=\gamma_j,\ \ j=\overline{1,m}. \tag{7}$$

Accept the following denotation

$$\Delta_{js} = \sum_{k=0}^{m-1} \lambda_s^k \left[\alpha_{jk} h_{\frac{1}{n}}(a,\lambda_s) + \beta_{jk} h_{\frac{1}{n}}(b,\lambda_s) \right], \ j = \overline{1,m}, \ s = \overline{1,m} . \tag{8}$$

Then the system of linear algebraic equations will take the following form

$$\sum_{s=1}^{m} \Delta_{js} C_s = \gamma_j , \ j = \overline{1,m} . \tag{9}$$

For applying the Kramer principle to this system, calculate its determinant and accept that determinant is non-zero

$$\Delta = \begin{vmatrix} \Delta_{11} \ldots\ldots\ldots \Delta_{1m} \\ \ldots\ldots\ldots\ldots\ldots \\ \Delta_{m1} \ldots\ldots\ldots \Delta_{mm} \end{vmatrix} \neq 0 . \tag{10}$$

Then from (9) we get

$$C_s = \frac{1}{\Delta} \begin{vmatrix} \Delta_{11} \ldots\ldots \Delta_{1s-1} & \gamma_1 & \Delta_{1s+1} \ldots\ldots \Delta_{1m} \\ \Delta_{21} \ldots\ldots \Delta_{2s-1} & \gamma_2 & \Delta_{2s+1} \ldots\ldots \Delta_{2m} \\ \ldots\ldots\ldots\ldots & \ldots & \ldots\ldots\ldots\ldots \\ \Delta_{m1} \ldots\ldots \Delta_{ms-1} & \gamma_m & \Delta_{ms+1} \ldots\ldots \Delta_{mm} \end{vmatrix} = \frac{1}{\Delta} \sum_{p=1}^{m} \gamma_p \Delta^{(p,s)} , \tag{11}$$

where Δ denotes an algebraic complement of the element standing in the intersection of the p -th line and the s -th column of the determinant Δ .

Substituting (11) in (6), we get the solution of boundary value problem in the following form

$$y(x) = \frac{1}{\Delta} \sum_{s=1}^{m} h_{\frac{1}{n}}(x,\lambda_s) \sum_{p=1}^{m} \gamma_p \Delta^{(p,s)} = \sum_{p=1}^{m} \left[\frac{1}{\Delta} \sum_{s=1}^{m} \Delta^{(p,s)} h_{\frac{1}{n}}(x,\lambda_s) \right] \gamma_p . \tag{12}$$

So, we get the following statement.

Theorem. If in the problem (1),(2), $a_m \neq 0$, the conditions (2) are linear independent, condition (10) is satisfied, then (12) is valid.

Remark 1. Let's consider the following fractional differential equation

$$a_1 D^{\frac{m_1}{n_1}} y(x) + a_2 D^{\frac{m_2}{n_2}} y(x) + \ldots + a_k D^{\frac{m_k}{n_k}} y(x) = \gamma(x)$$

$$\sum_{s=1}^{k} a_s D^{\frac{m_s}{n_s}} y(x) = \gamma(x), \tag{13}$$

where each order derivative is a fraction. If we find a common denominator for these fractions, then equation (3) (if the common denominator is n) by means of the step $\frac{1}{n}$ may be reduced to the equation of the form (1).

Remark 2. In the considered equation (2), instead of the step $\frac{1}{n}$ we can take any positive real number. This time, instead of fractional order differential equation we get a boundary value problem for an irrational order differential equation.

Remark 3. In the case under consideration, as the constructed, as the constructed invariant functions contain negative degrees of arbitrary variable (as Mittag-Laffler functions), we accept that the point x doesn't enter into the considered interval.

Remark 4. The considered problem is an analogy of the boundary value problem for a fractional order ordinary constant coefficient linear differential equation stated for an ordinary constant coefficient linear differential equation.

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Kəsr tərtib törəməli sabit əmsallı adi xətti diferensial tənlik üçün sərhəd məsələsi

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XÜLASƏ

Bu işdə kəsr tərtib törəməli adi, xətti, sabit əmsallı diferensial tənlik üçün qeyri-lokal sərhəd şərti daxiiində məsələyə baxılmışdır.

Kəsr tərtib törəməyə nəzərən invariant həllin köməyi ilə baxılan tənliyin asılı olmayan həlləri qurulmuş, sonra isə alınan asılı olmayan həllərin xətti kombinasiyası kimi ümumi həll qurulmuşdur.

Ümumi həllə daxil olan sabitlər (onların sayi sərhəd şərtləri qədərdir) verilmiş sərhəd şərtlərindən təyin edilmişdir. Bununla da sərhəd məsələsinin həlli analitik şəkildə qurulmuşdur.

Açar sözlər: kəsr törəmə, Liuvill tərifi, Mittaq- Leffler funksiyaları, kəsr törəməyə nəzərən invariant.

Граничная задача для обыкновенного линейного дифференциального уравнения дробного порядка с постоянными коэффициентами

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РЕЗЮМЕ

Работа посвящена исследованию решения граничной задачи для обыкновенного линейного дифференциального уравнения дробного порядка.

Исследование приводится исходя из инварианта полученного для дробной производной.

Строится общее решение рассматриваемого уравнения, постоянные входящиеся в общее решение (их число совпадает с числом граничных условий), определяются из заданных граничных условий. С этим определяется аналитический вид решения рассматриваемой граничной задачи.

Ключевые слова: дробная производная, определение Лиувилля, функции Миттаг-Леффлера, инвариант дробной производной.